

# Entropy and Mass Bounds of Kerr-de Sitter Spacetimes

A.M. Ghezelbash<sup>1</sup> and R. B. Mann<sup>2</sup>

Department of Physics, University of Waterloo,  
Waterloo, Ontario N2L 3G1, Canada

## Abstract

We consider Kerr-de Sitter spacetimes and evaluate their mass, angular momentum and entropy according to the boundary counterterm prescription. We provide a physical interpretation for angular velocity and angular momentum at future/past infinity. We show that the entropy of the four-dimensional Kerr-de Sitter spacetimes is less than of pure de Sitter spacetime in agreement to the entropic N-bound. Moreover, we show that maximal mass conjecture which states “*any asymptotically de Sitter spacetime with mass greater than de Sitter has a cosmological singularity*” is respected by asymptotically de Sitter spacetimes with rotation. We furthermore consider the possibility of strengthening the conjecture to state that “*any asymptotically dS spacetime will have mass greater than dS if and only if it has a cosmological singularity*” and find that Kerr-de Sitter spacetimes do not respect this stronger statement. We investigate the behavior of the  $c$ -function for the Kerr-de Sitter spacetimes and show that it is no longer isotropic. However an average of the  $c$ -function over the angular variables yields a renormalization group flow in agreement with the expansion of spacetime at future infinity.

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<sup>1</sup>EMail: [amasoud@sciborg.uwaterloo.ca](mailto:amasoud@sciborg.uwaterloo.ca)

<sup>2</sup>EMail: [mann@avatar.uwaterloo.ca](mailto:mann@avatar.uwaterloo.ca)

# 1 Introduction

There is still much to be learned about holographic duality, particularly for spacetimes that are asymptotically de Sitter (dS). Amongst the many challenges presented in establishing (or perhaps refuting) a dS holographic correspondence principle is the calculation of conserved quantities. Unlike their asymptotically flat or asymptotically anti de Sitter counterparts, asymptotically de Sitter spacetimes have neither a spatial infinity nor a global timelike Killing vector. Consequently both the definition and the computation of conserved charges for such spacetimes are not straightforward.

However it has recently been shown that it is possible under certain circumstances to compute such charges at past or future infinity [1]. This method, analogous to the Brown-York prescription in asymptotically AdS spacetimes [2, 3, 4, 5], is inspired from a conjectured holographic duality between the physics in asymptotically dS spacetimes and that of a boundary Euclidean conformal field theory (CFT) that resides on past or future infinity. The specific prescription in ref. [1] (employed previously by others in more restricted contexts [6, 7]) can be generalized to an arbitrary number of dimensions, in which there is an algorithmic prescription for adding boundary terms that render the action finite [8, 9].

Carrying out a procedure analogous to that in the AdS case [2, 4], it is straightforward to compute the boundary stress tensor, and from this obtain conserved charges associated with the asymptotically dS (adS) spacetime at future/past infinity. In particular the conserved charge associated with the (asymptotic) Killing vector  $\partial/\partial t$  – now spacelike outside of the cosmological horizon – can be interpreted as the conserved mass. Employing this definition, the authors of ref. [1] were led to the conjecture that *any asymptotically dS spacetime with mass greater than dS has a cosmological singularity*. We shall refer to this conjecture as the maximal mass conjecture. As stated the conjecture is in need of clarification before a proof can be considered, but roughly speaking it means that the conserved mass of any physically reasonable adS spacetime must be negative (i.e. less than the zero value of pure dS spacetime). This has been verified for topological dS solutions and their dilatonic variants [10] and for Schwarzschild-de Sitter (SdS) black holes up to dimension nine [8]. The maximal mass conjecture was based in part on the Bousso N-bound [11], another conjecture stating that *any asymptotically dS spacetime will have an entropy no greater than the entropy  $\pi\ell^2$  of pure dS with cosmological constant  $\Lambda = 3/\ell^2$  in (3+1) dimensions*.

Recently it has been shown that locally asymptotically dS spacetimes with NUT charge furnish counterexamples to both of these conjectures [12, 13]. Specifically, there is a range of parameter space in which the conserved mass of the NUT-charged dS spacetime can be greater than zero, and/or its entropy exceeds that of pure dS spacetime. Note that outside of the cosmological horizon NUT-charged dS spacetimes are not afflicted with closed timelike curves the way that their asymptotically flat and anti de Sitter counterparts are.

In this paper we extend these considerations to charged dS spacetimes with rotation. There are several reasons for being interested in this class of spacetimes. One is to understand the physical interpretation of angular momentum outside of the horizon as well as the role and the contribution

of the rotation to the entropy. Another is to understand the nature of the  $c$ -function [14] when rotation is present. We find that in general it becomes a function of angle, and we describe how to physically interpret it in this more general setting. Yet another reason is to explore the validity and utility of the maximal mass conjecture when rotation is present. One might hope that the maximal mass conjecture, if correct, might serve as a diagnostic tool in determining the presence/absence of singularities. This would be the case if the conjecture could be strengthened to state that *any asymptotically dS spacetime will have mass greater than dS if and only if it has a cosmological singularity*. This stronger version of the conjecture would then imply that a spacetime with negative mass at future infinity would be free of cosmological singularities.

To this end we shall consider in this paper spacetimes with zero NUT charge, deferring the investigation of rotating asymptotically dS spacetimes with non-vanishing NUT charge to future work [15]. We proceed as follows. We consider the Kerr-dS spacetime and review the procedure for calculating the conserved mass, angular momentum and entropy of the spacetime. We then examine the behavior of different physical quantities of the spacetime and provide a physical interpretation of quantities such as angular velocity and angular momentum at future (past) infinity. We find that the N-bound and maximal mass conjectures are never violated even when the black hole is extremal. However we also find that Kerr-dS spacetime can have naked singularities even when the mass (at future infinity) is negative. Consequently it is difficult to see how the stronger version of the conjecture can be realized for any spacetimes of physical interest. We then extend the notion of the  $c$ -function [14] to this situation and show that it depends on the angular coordinate, due to rotation. We then average the  $c$ -function over the angular variables and show that the renormalization group flow is in agreement with the expansion of spacetime at future infinity.

## 2 Kerr-dS spacetime

The Euclidean Kerr-dS geometry is given by the line element

$$ds^2 = \frac{\Delta_E(r)}{\Xi_E^2 \rho_E^2} (dt + a \sin^2 \theta d\phi)^2 + \frac{\Theta_E(\theta) \sin^2 \theta}{\Xi_E^2 \rho_E^2} [adt - (r^2 - a^2) d\phi]^2 + \frac{\rho_E^2 dr^2}{\Delta_E(r)} + \frac{\rho_E^2 d\theta^2}{\Theta_E(\theta)} \quad (1)$$

where

$$\begin{aligned} \rho_E^2 &= r^2 - a^2 \cos^2 \theta \\ \Delta_E(r) &= -\frac{r^2(r^2 - a^2)}{\ell^2} + r^2 - 2mr - a^2 \\ \Theta_E(\theta) &= 1 - \frac{a^2}{\ell^2} \cos^2 \theta \\ \Xi_E &= 1 - \frac{a^2}{\ell^2} \end{aligned} \quad (2)$$

For  $a \leq \ell$ , the metric function  $\Theta_E(\theta)$  always is positive and the Euclidean section exists for some values of the coordinate  $r$  between different successive roots of  $\Delta_E(r)$ , such that  $\Delta_E(r)$  is a positive valued function there. Depending on the values of the rotation parameter  $a$ , the cosmological

parameter  $\ell = \sqrt{3/\Lambda}$  (where  $\Lambda$  is the cosmological constant) and the mass parameter  $m$ , the function  $\Delta_E(r)$  can have between two to four real roots.

The Lorentzian geometry is given by

$$ds^2 = -\frac{\Delta_L(r)}{\Xi_L^2 \rho_L^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Theta_L(\theta) \sin^2 \theta}{\Xi_L^2 \rho_L^2} [adt - (r^2 + a^2) d\phi]^2 + \frac{\rho_L^2 dr^2}{\Delta_L(r)} + \frac{\rho_L^2 d\theta^2}{\Theta_L(\theta)} \quad (3)$$

where

$$\begin{aligned} \rho_L^2 &= r^2 + a^2 \cos^2 \theta \\ \Delta_L(r) &= -\frac{r^2(r^2+a^2)}{\ell^2} + r^2 - 2mr + a^2 \\ \Theta_L(\theta) &= 1 + \frac{a^2}{\ell^2} \cos^2 \theta \\ \Xi_L &= 1 + \frac{a^2}{\ell^2} \end{aligned} \quad (4)$$

The event horizons of the spacetime are given by the singularities of the metric function, which are the real roots of  $\Delta_L(r) = 0$ . Hence the horizons are determined by the solutions of the equation

$$r_H^4 - r_H^2(\ell^2 - a^2) + 2m\ell^2 r_H - \ell^2 a^2 = 0 \quad (5)$$

Note that the roots of this equation are not the same as those of the function  $\Delta_E(r)$ . The smallest root of  $\Delta_L(r)$  is negative, while the other three roots corresponding to inner, outer and cosmological horizons are positive. The Lorentzian metric function  $\Delta_L(r) \geq 0$  between its smallest root and the inner horizon and between the outer and cosmological horizons, where  $t$  is a timelike coordinate. It is negative elsewhere, where  $t$  is spacelike.

In the limit  $\ell \rightarrow \infty$ , equation (5) yields the well known location of the Kerr black hole horizon

$$r_H = m + \sqrt{m^2 - a^2} \quad (6)$$

When the rotational parameter is very small  $a \rightarrow 0$ , then the equation (5) reduces to

$$\frac{r_H^3}{\ell^2} - r_H + 2m = 0 \quad (7)$$

which gives us the location of Schwarzschild-dS event horizon  $r_H$  and cosmological horizon  $r_C$ . In this case, for mass parameters  $m$  with  $0 < m < m_N$ , where

$$m_N = \frac{\ell}{3\sqrt{3}} \quad (8)$$

we have a black hole in dS spacetime with event horizon at  $r = r_H$  and cosmological horizon at  $r = r_C > r_H$ . When  $m = m_N$ , the event horizon coincides with the cosmological horizon  $r_C = r_H = \frac{\ell}{\sqrt{3}}$  and one gets the rotating Nariai solution. For  $m > m_N$ , the Schwarzschild-dS metric describes a naked singularity in an asymptotically dS spacetime. So demanding the absence of naked singularities yields an upper limit to the mass of the Schwarzschild-dS black hole. The

other extreme case is when  $a \rightarrow \infty$  (i.e. negligible  $m$ ), which can straightforwardly be shown to be pure dS. Eq. (5) gives

$$r_H = \ell \quad (9)$$

for the horizon.

In general for the metric (3), the rotating Nariai solution has an event horizon (coincident with its cosmological horizon) at

$$r_H = \frac{3m + \sqrt{9m^2 - 8a^2(1 - \frac{a^2}{\ell^2})}}{2(1 - \frac{a^2}{\ell^2})} \quad (10)$$

which reduces to  $r_H = \frac{\ell}{\sqrt{3}}$  when  $a = 0$ . For a fixed value of  $a$ , we can find from (5) the following equations for the extremum of the horizon radius

$$2r_H + m \mp \sqrt{m^2 + 8mr_H} - 2r_H^3/\ell^2 = 0 \quad (11)$$

for which the upper branch has a maximum at  $r_+ = (\frac{3}{4} + \frac{\sqrt{3}}{2})m$  for fixed  $\ell$ . The corresponding critical cosmological parameter is  $\ell_+ = \frac{m}{4}(2\sqrt{3} + 3)^{3/2}$ . In this case from eq. (5), we find that  $r_+$  is a triple root and the corresponding rotational parameter is  $a = \frac{m}{4}\sqrt{6\sqrt{3} + 9}$ . This case is a special case of general rotating Nariai solution where all three horizons coincide to each other.

Other interesting special cases of the Kerr-dS metric are the cold metrics (where the inner and outer horizons coincide) and the lukewarm metrics (where the surface gravities of the cosmological and outer horizons are the same). A full analysis of these cases has been carried out previously [16] and we shall not reproduce these results here.

Denoting the four real roots (which collectively sum to zero) of  $\Delta_L(r)$  in increasing order by  $-c - \alpha, -c + \alpha, c - \beta, c + \beta$ , where  $0 \leq \beta < c$  and  $c < \alpha \leq 2c - \beta$ , we see that the first root is negative and that the other three roots corresponding to inner, outer and cosmological horizons are positive. In the limit  $\beta \rightarrow 0$ , we have a rotating Nariai solution [16] for which the metric becomes

$$ds_{Nariai}^2 = -\tilde{\Delta}_L(\tilde{r})\tilde{\rho}_L^2 d\tau^2 + \frac{\Theta_L(\theta) \sin^2 \theta}{\tilde{\rho}_L^2} [2ac\tilde{r}d\tau + \frac{c^2 + a^2}{\Xi_L} d\varphi]^2 + \tilde{\rho}_L^2 \left( \frac{d\tilde{r}^2}{\tilde{\Delta}_L(\tilde{r})} + \frac{d\theta^2}{\Theta_L(\theta)} \right) \quad (12)$$

where the new coordinates  $\tilde{r}, \varphi$  and  $\tau$  are given by

$$\begin{aligned} \tilde{r} &= \frac{r-c}{\beta} \\ \varphi &= \phi - \frac{a}{a^2+c^2}t \\ \tau &= \frac{\beta}{(a^2+c^2)\Xi_L}t \end{aligned} \quad (13)$$

and  $\tilde{\Delta}_L(\tilde{r}) = \frac{1}{\ell^2}(2c - \alpha)(2c + \alpha)(1 - \tilde{r}^2)$ ,  $\tilde{\rho}_L^2 = c^2 + a^2 \cos^2 \theta$ . Note that the metric (12) is a special case (i.e. no electric and magnetic charges) of the rotating Nariai solution for Kerr-dS spacetimes

with electric and magnetic charges [16]. In the general case, the solution is given by the metric (12) and an electromagnetic potential. In the limiting case  $a \rightarrow 0$ , the solution reduces to the non-rotating charged Nariai solution considered in [17].

The Nariai solution is in thermal equilibrium due to the coincidence of the outer and cosmological horizons, with common temperature  $T_{Nariai} = \frac{1}{4\pi\ell^2}(4c^2 - \alpha^2)$ . In general, the non-extremal (electric and/or magnetic charged) Kerr-dS black holes are not in thermal equilibrium since the temperatures of the outer and cosmological horizons are not the same. The exceptions to this rule are the Nariai solution and the lukewarm solution, that is the solution in which the cosmological and outer black hole horizons remain distinct, but have identical surface gravities. This occurs when the rotational parameter takes on the particular values given by the relation  $a^2 = c^2 - \frac{\alpha^2 + \beta^2}{2}$  [16].

Outside of the cosmological horizon, the Kerr-dS metric function  $\Delta_L(r)$  is negative, so we set  $r = \tau$  and rewrite the line element in the form

$$ds^2 = -\frac{\rho^2 d\tau^2}{\Delta(\tau)} + \frac{\Delta(\tau)}{\Xi^2 \rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Theta(\theta) \sin^2 \theta}{\Xi^2 \rho^2} [adt - (\tau^2 + a^2) d\phi]^2 + \frac{\rho^2 d\theta^2}{\Theta(\theta)} \quad (14)$$

where

$$\begin{aligned} \rho^2 &= \tau^2 + a^2 \cos^2 \theta \\ \Delta(\tau) &= \frac{\tau^2(\tau^2 + a^2)}{\ell^2} - \tau^2 + 2m\tau - a^2 \\ \Theta(\theta) &= \Theta_L(\theta) \\ \Xi &= \Xi_L \end{aligned} \quad (15)$$

The angular velocity of the horizon is given by

$$\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}} \Big|_{\tau=\tau_c} = \frac{a}{\tau_c^2 + a^2} \quad (16)$$

where  $\tau_c$  is the cosmological horizon ( $\Delta(\tau_c) = 0$  and  $\Delta(\tau > \tau_c) > 0$ ). The Killing vector  $\chi^\mu = \zeta^\mu + \Omega_H \psi^\mu$  is normal to the cosmological horizon  $\tau = \tau_c$ , where  $\zeta^\mu = (0, 1, 0, 0)$  is the stationary Killing vector and  $\psi^\mu = (0, 0, 0, 1)$  is the axial Killing vector with respect to coordinate system  $(\tau, t, \theta, \phi)$ . The surface gravity of the black hole on the horizon is given by  $\kappa = \sqrt{\frac{-1}{2} \nabla^\mu \chi^\nu \nabla_\mu \chi_\nu}$  [18] which becomes

$$\kappa = \frac{1}{2(\tau_c^2 + a^2)\Xi} \frac{d\Delta}{d\tau} \Big|_{\tau=\tau_c} \quad (17)$$

To compute the conserved mass and the entropy of the spacetime outside the cosmological horizon, we consider the four dimensional action that yields the Einstein equations with a positive cosmological constant

$$I = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} (R - 2\Lambda) - \frac{1}{8\pi} \int_{\partial\mathcal{M}^-} d^3x \sqrt{h^\pm} K^\pm + I_{ct} \quad (18)$$

where  $\partial\mathcal{M}^\pm$  are the future/past boundaries, and  $\int_{\partial\mathcal{M}^-}^{\partial\mathcal{M}^+} d^3x$  indicates an integral over a future boundary minus an integral over a past boundary, with respective induced metrics  $h_{\mu\nu}^\pm$  and intrinsic/extrinsic curvatures  $K_{\mu\nu}^\pm$  and  $\hat{R}(h^\pm)$  induced from the bulk spacetime metric  $g_{\mu\nu}$ .  $I_{ct}$  is the counter-term action, calculated to cancel the divergences from the first two terms (given in [8]) and we set the gravitational constant  $G = 1$ . The associated boundary stress-energy tensor is obtained by the variation of the action with respect to the boundary metric, the explicit form of which can be found in [8].

If the boundary geometries have an isometry generated by a Killing vector  $\xi^\pm$ , then it is straightforward to show that  $T_{ab}^\pm \xi^{\pm b}$  is divergenceless, from which it follows that there will be a conserved charge  $\mathfrak{Q}^\pm$  between surfaces of constant  $t$ , whose unit normal is given by  $n^{\pm a}$ . Physically this means that a collection of observers on the hypersurface all observe the same value of  $\mathfrak{Q}$  provided this surface had an isometry generated by  $\xi^b$  (for explicit calculations, see [8]). If  $\partial/\partial t$  is itself a Killing vector, then this can be defined as  $\mathfrak{Q}^\pm = \mathfrak{M}^\pm$ , the conserved mass associated with the future/past surface  $\Sigma^\pm(\tau)$  at any given point  $t$  on the boundary. This quantity changes with the cosmological time  $\tau$ . Since all asymptotically dS spacetimes must have an asymptotic isometry generated by  $\partial/\partial t$ , there is at least the notion of a conserved total mass  $\mathfrak{M}^\pm$  for the spacetime in the limit that  $\Sigma^\pm$  are future/past infinity.

### 3 Action, Mass and Entropy

For the Killing vector  $\zeta^\mu$ , which is spacelike outside the cosmological horizon, we obtain the conserved mass

$$\mathfrak{M} = -\frac{m}{\Xi^2} + \frac{4a^4 + 5\ell^4}{40\ell^2\Xi^2} \frac{1}{\tau} + O\left(\frac{1}{\tau^2}\right) \quad (19)$$

for the metric (14) near future infinity. This result holds even for the charged Kerr-dS spacetimes [19].

We note that for  $a = 0$  (and  $\Xi = 1$ ), the total mass (19) reduces exactly to the total mass of the four dimensional Schwarzschild-dS black hole [8]. We observe that for all the Kerr-dS black holes with positive mass parameter, the total mass is negative, satisfying the maximal mass conjecture [1] since the spacetime (14) is free of any cosmological singularities. The total action (18) of the spacetime is

$$I = -\frac{\beta_H(\tau_c^3 + a^2\tau_c + m\ell^2)}{2\ell^2\Xi^2} + \frac{\beta_H(4a^4 + 20a^2\ell^2 + 15\ell^4)}{120\ell^2\Xi^2} \frac{1}{\tau} + O\left(\frac{1}{\tau^2}\right) \quad (20)$$

where  $\beta_H$  is the analogue of the Hawking temperature outside of the cosmological horizon. It is related to the surface gravity of the horizon  $\kappa$  by

$$\beta_H = \left| \frac{2\pi}{\kappa} \right| = \frac{4\pi(\tau_c^2 + a^2)\Xi}{\left| \frac{d\Delta}{d\tau} \right|_{\tau=\tau_c}} = \frac{2\pi(\tau_c^2 + a^2)(\ell^2 + a^2)}{|2\tau_c^3 + \tau_c(a^2 - \ell^2) + m\ell^2|} \quad (21)$$

We note that in the limit  $a \rightarrow 0$ , the above  $\beta_H$  approaches  $\frac{2\pi\tau_c^2\ell^2}{2\tau_c^3 - \tau_c\ell^2 + m\ell^2} = \frac{2\pi\tau_c^2\ell^2}{\tau_c^3 - m\ell^2}$  where we use the relation  $\tau_c\ell^2 = \tau_c^3 + 2m\ell^2$  for the location of Schwarzschild-dS cosmological horizon, to eliminate

the linear term of  $\tau_c$  in the denominator. The last quantity is in exact agreement with the analogue of the Hawking temperature outside of the cosmological horizon of the Schwarzschild-dS black hole [8]. The other conserved charge, associated with axial Killing vector  $\psi^\mu = (0, 0, 0, 1)$  is the angular momentum

$$\mathfrak{J} = -\frac{am}{\Xi^2} \quad (22)$$

which in the limiting case of Schwarzschild-dS ( $a = 0$ ) is zero.

Extending the definition of entropy to asymptotically dS spacetimes [8, 13] with rotation, we employ the relation  $S = \lim_{\tau \rightarrow \infty} (\beta_H \mathcal{M} - I)$  where  $\mathcal{M}$  is the conserved charge associated with the Killing vector  $\chi^\mu$ . Note that  $\mathcal{M}$  is distinct from the mass  $\mathfrak{M}$  in eq. (19). We find for the entropy

$$S = \frac{\pi(\tau_c^2 + a^2)}{\Xi} \quad (23)$$

which is exactly 1/4 of the area of the cosmological horizon. We note that the following relations

$$\begin{aligned} \left(\frac{\partial S}{\partial \mathfrak{M}}\right)_{\mathfrak{J}} &= \widehat{\beta}_H \\ \left(\frac{\partial S}{\partial \mathfrak{J}}\right)_{\mathfrak{M}} &= -\widehat{\beta}_H \widehat{\Omega}_H \end{aligned} \quad (24)$$

are satisfied with our result obtained in equations (16), (21), (22) and (23) where  $\widehat{\beta}_H = \frac{\beta_H}{\Xi}$  and  $\widehat{\Omega}_H = \Omega_H \Xi - \frac{a}{\ell^2}$ . So we find that the first law of thermodynamics

$$d\mathfrak{M} = \frac{1}{\widehat{\beta}_H} dS + \widehat{\Omega}_H d\mathfrak{J} \quad (25)$$

is valid, where  $\widehat{\Omega}_H = \frac{a(\ell^2 - \tau_c^2)}{\ell^2(\tau_c^2 + a^2)}$  is the angular velocity of the horizon.

Since the normal at past/future infinity is timelike and not spacelike some care must be taken in physically interpreting this quantity. For the Kerr and Kerr-AdS spacetimes, the closest notion one has to a family of static observers outside of a black hole is that of locally non-rotating observers, whose coordinate angular velocity is given by  $\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}}$ . In a stationary spacetime this will be a function of radial position, diminishing to either zero (for Kerr) or a finite value (for Kerr-AdS), which corresponds to an observer's angular velocity at infinity. However the situation for Kerr-dS is that of a collection of observers outside of a cosmological horizon, where  $\partial/\partial t$  is spacelike. The notion of angular velocity becomes that of a helical path that a given observer traces as he/she moves along curves at future (past) infinity whose tangent vectors are  $\partial/\partial t$ . In this case the  $\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} \rightarrow \Omega_\infty = \frac{a}{\ell^2 + a^2}$  for large  $\tau$ ; this corresponds to the rate of change of the angular position of the frame of reference with respect to the  $t$ -direction. Similarly, the angular momentum is conserved from place to place anywhere along curves at future (past) infinity whose tangent vectors are  $\partial/\partial t$ : a collection of observers on a constant- $t$  hypersurface (which itself does not enclose any bulk space [20]) all observe the same value of  $\mathfrak{J}$  regardless of the value of  $t$ . We shall continue to use the terms "angular velocity", "angular momentum", and "rotation at

infinity” keeping in mind this distinction in physical interpretation relative to the asymptotically flat and AdS cases.

The quantity  $\widehat{\Omega}_H$  is measured with respect to a frame that is not rotating at infinity, and differs from  $\Omega_H$ , which is the corresponding quantity with respect to a frame that is rotating at infinity with angular velocity  $\Omega_\infty$ . It is the angular velocity  $\widehat{\Omega}_H$  that appears in the first law, and is related to  $\Omega_H$  by  $\widehat{\Omega}_H/\Xi = \Omega_H - \Omega_\infty$ . The satisfaction of the first law of thermodynamic for Kerr-dS spacetimes with given conserved mass (19), angular momentum (22) and entropy (23) guarantees that the ratio of the entropy to the horizon area has the expected value of 1/4. The importance of using a frame that is not rotating at infinity has previously been noted for Kerr-AdS black holes [21], and a discussion of its relevance for the first law of thermodynamics has recently appeared [22]. What is interesting here is that the same distinction is required, despite the difference in physical interpretation in the asymptotically dS case.

Figure (1) shows the typical behavior of  $S$  as a function of rescaled mass  $m/\ell$  and rotational parameter  $a/\ell$ . We note that when  $a = 0$ , the cosmological horizon is located at  $\tau_c = 0$  for all values of  $m$  and the entropy is  $S = 0$ . By increasing the rotational parameter for a fixed  $m$ , the entropy monotonically increases and approaches the N-bound. As is clear from figure (1) in the special case of  $m = 0$ , for all values of  $a$  the entropy is equal to the N-bound. This is as expected since the  $m = 0$  metric is just pure dS spacetime in unusual coordinates. Note also the behavior of the entropy as the black hole approaches extremality. The mass and rotational parameters of the extremal black hole are given by

$$\begin{aligned} m_{ext.} &= \frac{4(2-\sqrt{3})}{3}\ell\sqrt{2\sqrt{3}-3} \\ a_{ext.} &= (2-\sqrt{3})\ell \end{aligned} \quad (26)$$

and in this case, the inner horizon and the outer horizon coincide with the cosmological horizon that is located at

$$\tau_{c(ext.)} = \frac{\sqrt{6\sqrt{3}-9}}{3}\ell \quad (27)$$

and the fourth root of the  $\Delta(\tau)$  (which is a negative and non-physical root) is  $-3\tau_{c(ext.)}$ . Applying the relation (23), shows that the entropy has a definite value of  $\pi\frac{9-5\sqrt{3}}{6(2-\sqrt{3})}\ell^2$  satisfying the N-bound.

In the other extremal case, where only the inner and outer horizons coincide, the cosmological horizon is given by  $\tau_{c(ext.)} = -\tau_{in(ext.)} + \frac{\ell^2-\tau_{in(ext.)}^2}{\sqrt{\ell^2+\tau_{in(ext.)}^2}}$  and the entropy always respects the N-bound for all the allowed values of the extremal inner horizon.

Consequently we confirm the expected result that the N-bound on entropy is satisfied for all the values of the mass and rotational parameter.

Figure (1) has several interesting features. For fixed  $m$  the entropy increases for increasing  $a$ , rapidly approaching the N-bound. For any value of  $a/\ell > 2$ , the entropy remains very close to the N-bound; in these situations  $m/a$  is relatively small, and the spacetime very close to pure dS. Although difficult to see from figure (1), the upper sheet slopes slightly downward toward the left of the diagram, so that by increasing the mass parameter  $m$ , for a fixed  $a$ , the entropy decreases and so departs more from the N-bound. Moreover, we see from eqs.(15) and (23) that in the limit

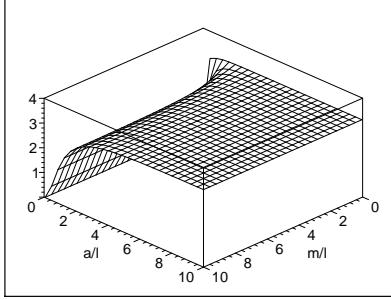


Figure 1: Entropy of Kerr-dS spacetime with respect to rescaled mass  $m/\ell$  and rotational parameter  $a/\ell$ . The horizontal sheet denotes the N-bound of  $\pi\ell^2$ .

$m \rightarrow 0$ , the entropy saturates the N-bound for all values of rotational parameter  $a$ . This is as expected since in the limit  $m \rightarrow 0$ , the Kerr-dS metric (14) reduces to the standard form of the dS spacetime under a coordinate transformation.

In both the AdS and dS cases there is a natural correspondence between phenomena occurring near the boundary (or in the deep interior) of either spacetime and UV (IR) physics in the dual CFT. Solutions that are asymptotically (locally) dS lead to an interpretation in terms of renormalization group flows and an associated generalized dS  $c$ -theorem. This theorem states that in a contracting patch of dS spacetime, the renormalization group flows toward the infrared and in an expanding spacetime, it flows toward the ultraviolet. In reference [23], a  $c$ -function was defined for a representation of the dS metric with a wide variety of boundary geometries involving direct products of flat space, the sphere and hyperbolic space. The definition of  $c$ -function in these cases, is based on its generalization from the flat boundary geometry. For a four-dimensional dS spacetime, it is given by

$$c = (G_{\mu\nu} n^\mu n^\nu)^{-1} \quad (28)$$

where  $n^\mu$  is the unit vector in the  $\tau$  direction. Consequently the  $c$ -theorem implies that the  $c$ -function (or  $\bar{c}$ -function) must increase (decrease) for any expanding (contracting) patch of the spacetime.

Since the spacetime (14) is asymptotically locally dS, if we use the relation (28), we find

$$c(\tau, \theta) = \frac{1}{3} \frac{\tau^4 + \tau^2(a^2 - \ell^2) + 2m\ell^2\tau - a^2\ell^2}{\tau^2 + a^2 \cos^2 \theta} \quad (29)$$

We see that, due to the presence of rotation, the  $c$ -function depends explicitly to the angular coordinate  $\theta$ . In the language of paper [23], this is due to the dependence of the effective cosmological constant on the angular variable. In figure (2), the  $c$ -functions for three different fixed angular coordinates, are plotted. We consider the following quantity

$$\bar{c}(\tau) = \frac{1}{\mathfrak{A}} \int_{\mathfrak{S}} c(\tau, \theta) dS \quad (30)$$

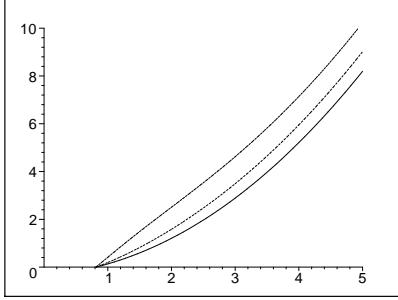


Figure 2: The  $c$ -function of Kerr-dS spacetime plotted as a function of  $\tau$  for three different values of angular variable  $\theta = 0$  (lower solid curve),  $\frac{\pi}{5}$  (dotted curve),  $\frac{\pi}{3}$  (upper dashed curve). The cosmological constant  $\Lambda = 3$  and the mass and rotational parameters are respectively set to 2 and 3. The cosmological horizon is located at  $\tau_c = 0.6449$ .

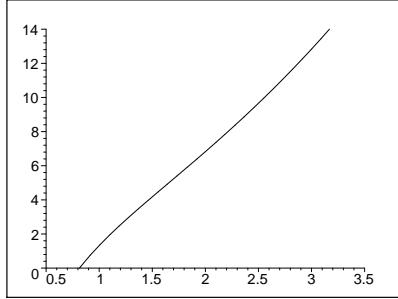


Figure 3: The  $\bar{c}$ -function of Kerr-dS spacetime plotted as a function of  $\tau$ . The cosmological constant  $\Lambda = 3$  and the mass and rotational parameters are respectively set to 2 and 3. The cosmological horizon is located at  $\tau_c = 0.6449$ .

which integration is over the 2-surface  $\mathfrak{S}$  (with area  $\mathfrak{A}$ ), parametrized by  $(\theta, \phi)$  for a fixed  $t$  and  $\tau$ , describing the average  $c$ -function of the spacetime (14). Although the  $c$ -function (and RG flow) is anisotropic, as the boundary is approached this anisotropy vanishes and the  $c$ -function (or  $\bar{c}$ -function) behaves  $\sim \tau^2$  in good agreement with one expects from the  $c$ -theorem. In figure (3), the  $\bar{c}$ -function outside the cosmological horizon, is plotted. In this plot, we set  $\ell = 1, m = 2$  and  $a = 3$ . The cosmological horizon is located at  $\tau_c = 0.6449$ .

As we see from the figure, outside the cosmological horizon for any value of  $\theta$ , the  $c$ -function is a monotonically increasing function of  $\tau$ , indicative of the expansion of a constant  $\tau$ -surface of the metric (14) outside of the cosmological horizon. The metric (14) at future infinity ( $\tau \rightarrow +\infty$ ) reduces to

$$-d\tau^2 + e^{2\chi/\ell} dF_3^2 \quad (31)$$

where  $\chi = \ell \ln \tau$ . Here  $dF_3^2$  is the metric of three-dimensional constant  $\chi$ -surface which is neither

maximally symmetric nor uniform in the  $\theta$ -direction. We note that the scale factor  $e^{2\omega/\ell}$  in (31) expands exponentially near future infinity, so the behavior of  $\bar{c}$ -function in figure (3) is in good agreement with what one expects from the  $c$ -theorem.

Although it seems that the  $c$ -function given in (29) is anisotropic in the limiting case of  $m \rightarrow 0$ , this is not the case. In fact in the limit  $m \rightarrow 0$ , by a coordinate transformations, the Kerr-dS metric (14) reduces to the following standard form of the pure dS spacetime

$$-(1 - \frac{y^2}{\ell^2})dT^2 + \frac{dy^2}{1 - \frac{y^2}{\ell^2}} + y^2(d\Theta^2 + \sin^2 \Theta d\psi^2) \quad (32)$$

for which the  $c$ -function is equal to the isotropic value  $\frac{y^2 - \ell^2}{3}$ . The angular dependence of the  $c$ -function in (29) is a consequence of the anisotropy of the metric functions in the coordinates we are using.

## 4 Conclusions

We have shown that the entropy and the mass of the class of Kerr-dS spacetimes always respect the N-bound and dS-maximal mass conjectures. These results hold even when the rotating hole approaches extremality. The first law of thermodynamics is also obeyed, albeit with differing interpretations of the physical quantities relative to their asymptotically flat and AdS counterparts.

For the dS spacetimes with rotation, we have shown that the notion of the  $c$ -function must be extended, describing an anisotropic renormalization group flow that becomes completely isotropic as the boundary is approached. The behavior of the averaged  $c$ -function is in complete agreement with the  $c$ -theorem.

Our results also provide substantive evidence against any stronger formulation of the maximal mass conjecture. Consider a Kerr-dS spacetime for sufficiently small mass and zero rotation. In this case there is a cosmological horizon, a negative mass at future infinity and an outer horizon censoring the singularity, in accord with the stronger version of the conjecture. For small nonzero rotation this situation is unchanged. But as the rotation parameter grows, the inner & outer horizons get closer together and eventually they coincide. For larger values of the rotation parameter, there is a naked singularity. However the mass at future infinity is still negative, ie still less than dS spacetime. For example, a spacetime with  $m = \ell/10$  and  $a = \ell/12$  does not have a cosmological (naked) singularities, whereas a spacetime with  $m = \ell/10$  and  $a = \ell/4$  does. Yet both have negative mass at future infinity. In general the spacetime will have a naked singularity with negative mass at future infinity whenever

$$\widehat{m} < \frac{\sqrt{6 - 6\widehat{a}^2 - 6\sqrt{(\widehat{a}^2 - 4\widehat{a} + 1)(\widehat{a}^2 + 4\widehat{a} + 1)}}}{18} \left( 2(1 - \widehat{a}^2) + \sqrt{(\widehat{a}^2 - 4\widehat{a} + 1)(\widehat{a}^2 + 4\widehat{a} + 1)} \right) \quad (33)$$

where  $\widehat{a} = \frac{a}{\ell}$  and  $\widehat{m} = \frac{m}{\ell}$ . Consequently there is a class of Kerr-dS spacetimes that have singularities, but still satisfy the (weaker) maximal mass conjecture. It is difficult to see how the conjecture could be strengthened in a manner that would render it useful for a significantly broad class of physically interesting spacetimes.

We conclude with a few comments about directions for future work. An obvious thing to consider are higher dimensional Kerr-dS spacetimes with multiple rotation parameters [24]; the behavior of the  $c$ -function in this case should exhibit a significantly greater degree of anisotropy. Another avenue to explore is the validity of the entropy-area relation  $S = A/4$ , which is satisfied for any black hole in a  $(d + 1)$  dimensional asymptotically flat/AdS setting, where  $A$  is the area of a  $d - 1$  dimensional fixed point set of isometry group. However, entropy can be defined for other kinds of spacetimes in which the isometry group has fixed points on surfaces of even co-dimension [25]. The best examples of these spacetimes are asymptotically locally flat and asymptotically locally anti dS spacetimes with NUT charge. In these cases when the isometry group has a two-dimensional fixed set (bolt), the entropy of the spacetime is not given by the area-entropy relation, as a consequence of the first law of thermodynamics [26].

In asymptotically dS spacetimes, the Gibbs-Duhem entropy is proportional to the area of the horizon and respects the N-bound (for the other case of Schwarzschild-dS spacetime, see [8]). However, for asymptotically locally dS spacetime with NUT charge, the entropy is no longer proportional to the area. Consequently the entropy need not respect the N-bound, an expectation confirmed by investigations for a wide range of situations with nonzero NUT charge [27, 28]. We are currently considering the validity of the N-bound and maximal mass conjectures for NUT charged asymptotically de Sitter spacetimes with rotation, and plan to report our results in a future publication [15].

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